Cultural imagery and statistical models of the force of mortality

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McGill University

2010.11.19

In order of appearance...

- 1950-1976: hazard function and incidence density
- 1711: allegorical essay, non-mathematical
- 1825 & 1832 : intensity/force of mortality
- 1897: imagery back to 1400's; mixtures of pdf's
- 2009: computer animations

Hazard rate / Hazard function / Force of mortality

- Central to modern survival analysis / epidemiology.
- Other names [Barlow (1963), Klein & Moeschberger (2003)]
  - Demography/Actuarial: "Force of mortality" $\longrightarrow$ life tables
  - Economics: "Mill's ratio" $\longrightarrow$ reciprocal for normal distr'n
  - Statistics: "Intensity function" $\longrightarrow$ extreme value theory
  - Statistics: "Conditional failure rate" $\longrightarrow$ reliability
  - Epidemiology: "Age-specific failure rate"
- Miettinen (1976): (short-term) "Incidence density"

Hazard rate: Definition

- Barlow et al. (1963)

$$\textit{cdf } F(x) \ \& \ \textit{pdf } f(x) \quad \rightarrow q(x) = \frac{f(x)}{1 - F(x)} \qquad [F(x) < 1].$$

- Wikipedia 2009: “limit of number of events per unit time divided by number at risk as time interval decreases”

$$h(t) = \lim_{\Delta t \to 0} \frac{\text{observed events}(t)/N(t)}{\Delta t}.$$

- Cox, 1972: “age specific failure rate”:

$$\lambda(t) = \lim_{\Delta t \to 0+} \frac{\text{pr}(t \leq T < t + \Delta t \mid t \leq T)}{\Delta t}.$$

Our edits of Wiki definition of hazard rate

- Omit somewhat contradictory '(t)' that follows word 'events'
- Make '#events' & '#s of persons at risk' more precise.
- Distinguish parameter (exp'd #) & statistic (obs'd #)
- Events occur within *interval* of width Δt
- $N(t) \rightarrow \overline{N}$, average # persons at risk during interval

$$\frac{\text{no. events in } (t, t+\Delta t)}{\overline{N} \times \Delta t} = \frac{\text{no. events in } (t, t+\Delta t)}{\text{Person-Time in } (t, t+\Delta t)}$$

takes form of incidence density, a term introduced to epidemiology by Miettinen (1976):

> *Incidence density ("force of morbidity" or "force of mortality") – perhaps the most fundamental measure of the occurrence of illness – is the number of new cases divided by the population-time (person-years of observation) in which they occur.*

Incidence density function $ID(t) \longleftrightarrow h(t)$ hazard function.

Who came up with the term 'hazard rate' ?

D.J. Davis. Rand Corp. An analysis of some failure data. JASA 1952 ???.

- "introduce[d] a new probability function which has been found useful in interpreting the physical causes of failure in terms of probability distributions."
- .. is termed the conditional density function of failure probability with time and is defined as the instantaneous probability rate of failure at time t conditional upon non-failure prior to time t" (p. 114).
- 'conditional density function'

$$Z(t) = f(t)/[1 - F(t)]\ ; \quad Z(t)dt = Pr\{t \leq T \leq t + dt \mid T \geq t\},$$

- "the actuarial concept of 'force of mortality' is precisely this conditional density function if a human being is considered as a system and death is defined as the failure of the system."
- failure models: machines/systems vs. 'human mortality': "[the function] for human mortality is similar in general characteristics to that of the normal theory except that in early life human mortality exhibits a non-zero conditional density.
- "This suggests the rationale that in youth, humans are subjected to a small death-hazard rate (force of mortality), but as they age they become increasingly weaker and, therefore, subject to an increasing death-hazard rate".

Early appearances of the term 'hazard rate'

- Zelen, Technometrics 1959:
  - Ref. to Davis (& "actuaries call it 'force of mortality' ")
  - First to have used the letter *h*. ???
  - .. [my] results good for [failure-time] distr'ns. having pdf of form $f(t) = h(t) \exp\{-\int_0^t h(x)dx\}$, where...

    "$h(t)$ is simply a non-negative function of time to failure t", and is sometimes referred to as the 'hazard', 'instantaneous rate of failure", or 'conditional failure rate'."

- Parzen text, Stochastic Processes 1962:
  - denotes it by $\mu(x)$
  - *"intensity fn. / hazard fn. / conditional rate of failure fn."*.
- Gaver Technometrics 1963 and Barlow Ann. Math. Stat. 1963,
  - First to have used word hazard in title of article ???
  - Gaver uses $\lambda(t)$ for hazard function and $H(t)$ for its integral.
  - Barlow et al. denote hazard function by $q(x)$

Visualizing the force of mortality: describing it to others

- Statisticians comfortable with concept of mathematical limit
- *Instantaneous* force of mortality more difficult for others
  - '0.00152 deaths per man-year at 40th birthday'
  - '0.00092 per woman-year at 40th birthday'
- How can we describe/visualise the force of mortality and quantities derived from it?
- How to give human form to components of $h(x) = \frac{f(x)}{1-F(x)}$?
  - 18th and 19th century teachers
  - 21st century computer animation

The force of mortality as seen historically

- Three British teachers
  - Joseph Addison(1672-1719)
    - early-18th century essayist/ poet/ politician
  - Benjamin Gompertz (1779-1865)
    - self-educated mathematician/ demographer / actuary
  - Karl Pearson (1857-1936)
    - historian/ 'Germanist'/ mathematician
    - founder of world's first university statistics department
- Visualize & represent force of mortality / derived quantities
  - allegories
  - physical and conceptual models
  - mathematics & statistical analysis

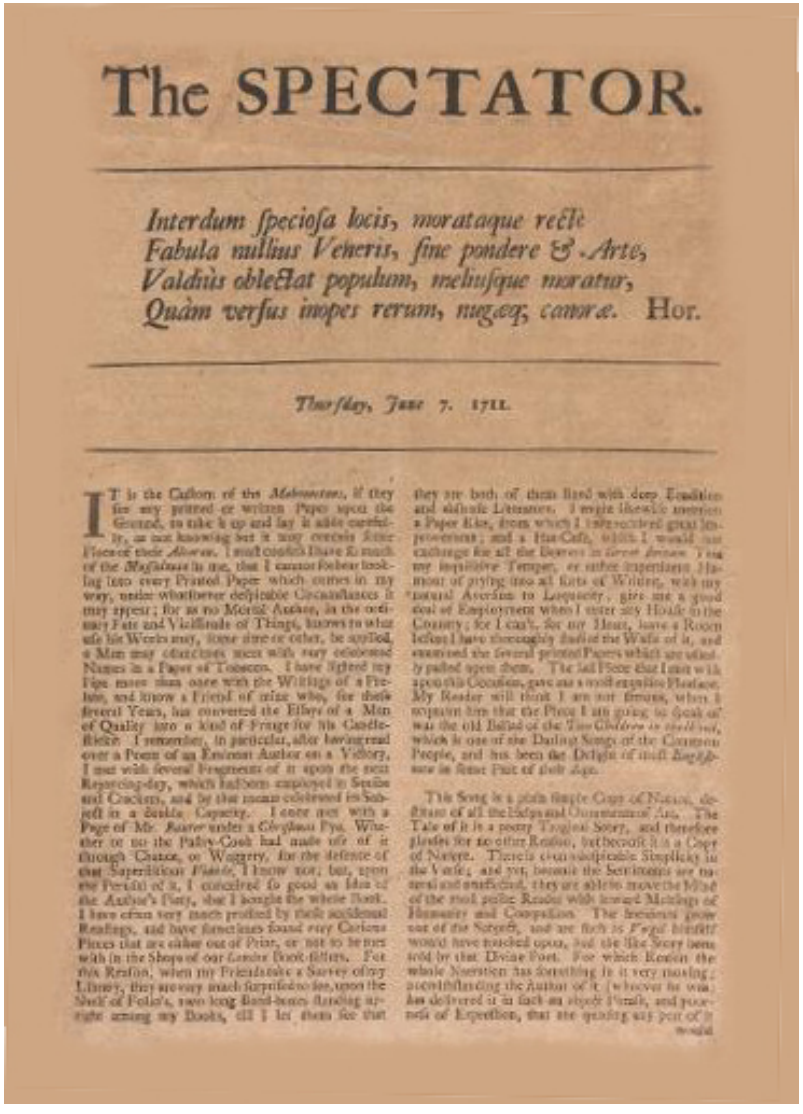
Force of mortality as seen by contemporary epidemiology teacher

Question: How do you visualize incidence density?

Miettinen (199x) :

- Army marches over bridge, and every so often a hole appears in random spot for short time & swallows up whoever happens to be marching over spot at that time.
- Army continues to travel over bridge in waves.
- Incidence density: number of persons swallowed up divided by, say, the total number of soldier-steps, or soldier-meters or soldier-minutes.
- Incidence density can vary as a function of distance from beginning of the bridge.

Force of mortality in *The Vision of Mirza*



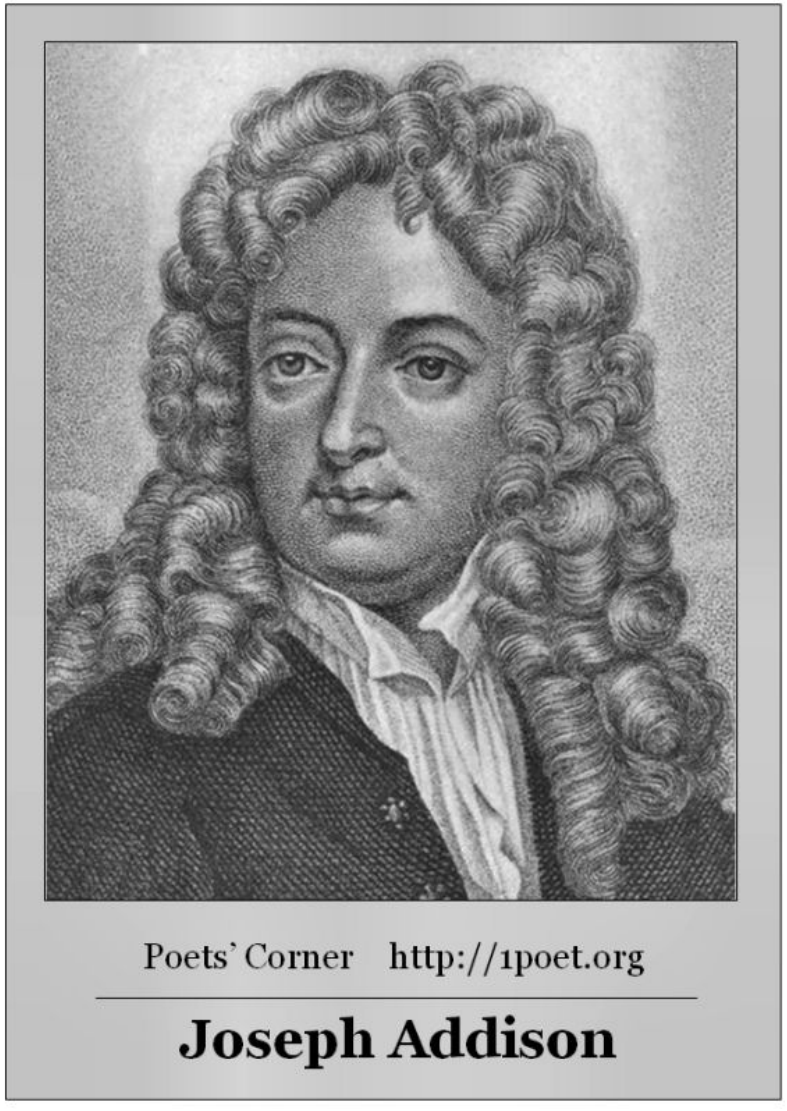
The SPECTATOR.

Interdum speciosa locis, morataque recte
Fabula nullius Veneris, sine pondere & Arte,
Valdius oblectat populum, meliusque moratur,
Quam versus inopes rerum, nugæque canoræ. Hor.

Thursday, June 7. 1711.

IT is the Custom of the *Mahometans*, if they see any printed or written Paper upon the Ground, to take it up and lay it aside carefully, as not knowing but it may contain some Verses of their *Alcoran*. I could wish I have so much of the *Mussulman* in me, that I cannot forbear looking into every Printed Paper which comes in my way, unless whatsoever despicable Circumstances it may appear; for as no Moral Action, in the ordinary Fate and Vicissitude of Things, knows to what ale his Words may, some time or other, be applied, a Man may sometimes meet with very celebrated Names in a Paper of Tobacco. I have lighted my Pipe more than once with the Writings of a Prelate, and know a Friend of mine who, for these several Years, has converted the Essays of a Man of Quality into a kind of Fringe for his Candlestick. I remember, in particular, after having read over a Poem of an Eminent Author on a Victory, I met with several Fragments of it upon the next Rejoicing-day, which happen'd employed in Serene and Crackers, and by that means celebrated in Sublime in a double Capacity. I once met with a Page of Mr. *Baxter* under a *Circulus* Pye. Whether or no the Pasty-Cook had made use of it through Chance, or Waggery, for the defence of our Superstitious *Viands*, I know not; but, upon the Perusal of it, I conceived so good an Idea of the Author's Piety, that I bought the whole Book. I have often very much profited by these accidental Readings, and have sometimes found very Curious Pieces that are either out of Print, or not to be met with in the Shops of our *London* Book-sellers. For this Reason, when my Friends take a Survey of my Library, they are very much surprised to see, upon the Shelf of Folio's, two long Band-boxes standing upright among my Books, till I let them see that they are both of them lined with deep Erudition and solemn Literature. I might likewise mention a Paper Kite, from which I have received great Improvement; and a Hat-Case, which I would not exchange for all the Beavers in *Great Britain*. I am my inquisitive Temper, or rather impertinent Humour of prying into all Sorts of Writing, with my natural Aversion to Laziness, give me a good deal of Employment when I enter my House in the Country; for I can't, for my Heart, leave a Room before I have thoroughly dipped the Walls of it, and examined the several printed Papers which are usually pasted upon them. The last Piece that I met with upon this Occasion, gave me a most agreeable Pleasure. My Reader will think I am not serious, when I acquaint him that the Piece I am going to speak of was the old Ballad of the *Two Children in the Wood*, which is one of the Darling Songs of the Common People, and has been the Delight of all *English*men in some Part of their Age.

This Song is a plain simple Copy of Nature, destitute of all the Helps and Ornaments of Art. The Tale of it is a pretty Tragical Story, and therefore pleases for no other Reason, but because it is a Copy of Nature. There is consequently a Simplicity in the Verse, and yet, because the Sentiments are natural and unaffected, they are able to move the Mind of the most polite Reader with several Motions of Humanity and Compassion. The Incidents grow out of the Subject, and are such as *Tragedy* itself would have touched upon, had the like Story been told by that Divine Poet. For which Reason the whole Narration has something in it very moving; notwithstanding the Author of it (whoever he was) has delivered it in such an abject Phrase, and poorness of Expression, that are quite sunk my part of it would



Poets' Corner http://1poet.org

Joseph Addison

(1672-1719)

Imagery in *The Vision of Mirza*

BRIDGE OF HUMAN LIFE

Multitudes of people passing over it

Passengers dropping through innumerable concealed trap-doors that...

- were set very thick at the entrance
- grew thinner towards the middle
- multiplied and lay closer together towards the end

Allegorical essay

Sept. 1, 1711

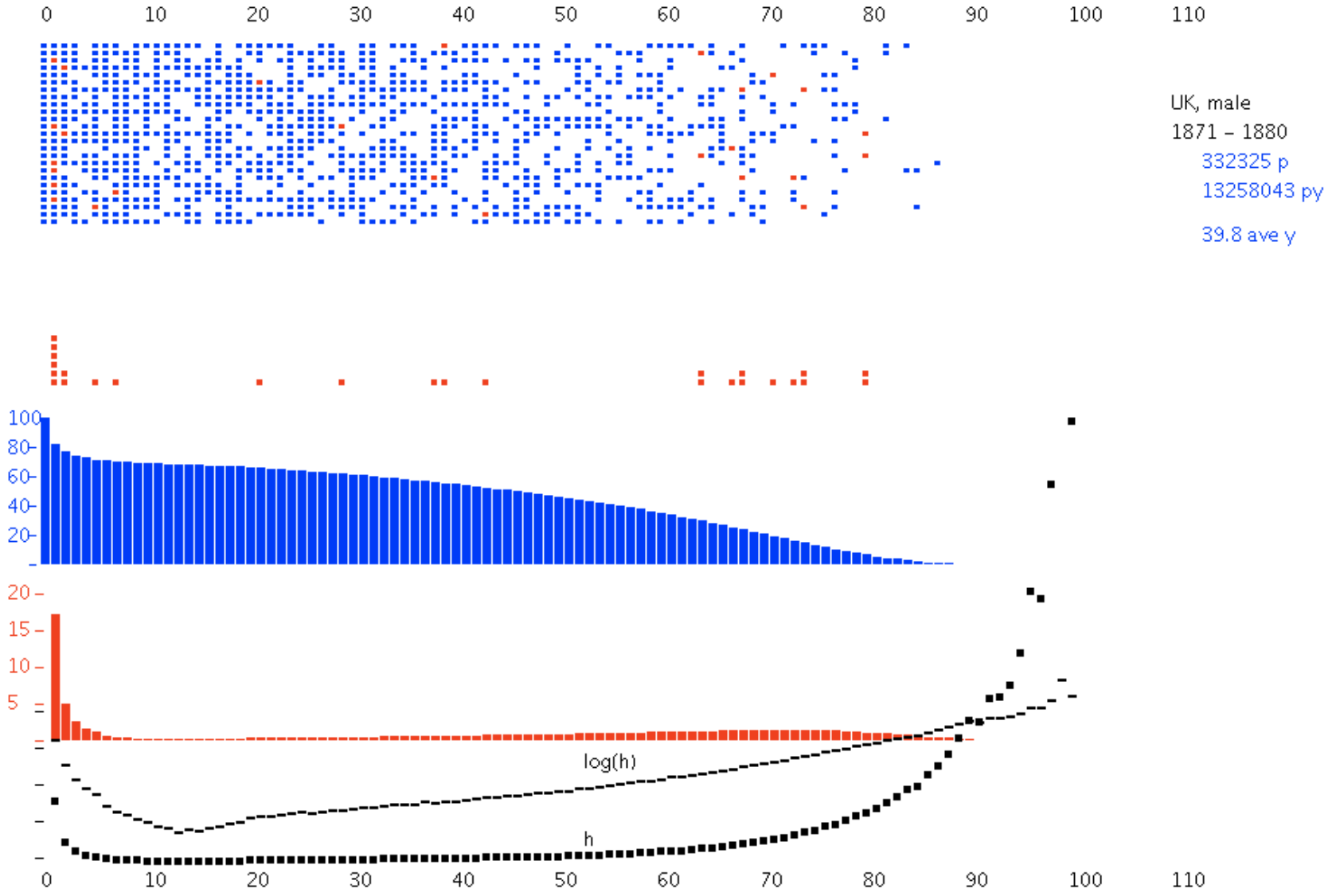
http://etc.usf.edu/clipart/17400/17410/mirza_17410.htm

Animated Bridge of Human Life

http://www.biostat.mcgill.ca/hanley/BridgeOfLife/

WARNING: Moving images shown

Animated Bridge of Human Life



http://www.biostat.mcgill.ca/hanley/BridgeOfLife/

Lifetables; statistical laws and regularity

Some contributors

1620-1674: John Graunt	'shopkeeper's arithmetic'
1629-1695: Christiaan Huygens	
1631-1699: Ludwig Huygens	
1656-1742: Edmund Halley	
1667-1754: Abraham DeMoivre	*
1687-1759: Nicholas Bernoulli	*
1700-1782: Daniel Bernoulli	*
1703-1768: Antoine Deparcieux	
1710-1761: Thomas Simpson	
1779-1865: Benjamin Gompertz	*
1803-1889: Thomas R Edmonds	*
1807-1883: William Farr	*
1826-1891: William Makeham	*

* mortality/survival rates as continuous mathematical functions.

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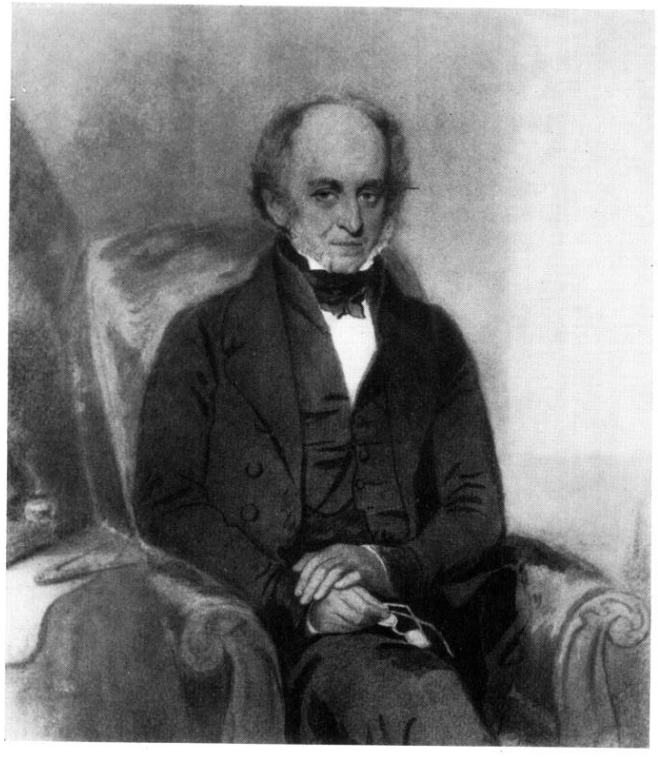
Gompertz' 'Law' of Mortality



BENJAMIN GOMPERTZ, 1779–1865

BENJAMIN GOMPERTZ. On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies. *Phil. Trans. R. Soc. London, 115. (1825), 513-583.*

" [his] paper [...] opened up a new approach to the life table. Previously, the table had been regarded as little more than a record of the number of persons surviving to successive integral ages out of a given number alive at an earlier age; Gompertz introduced the idea that l_x [the survival function] was a function connected by a mathematical relationship with a continuously operating force of mortality."

P. F. Hooker. J. Inst. Actuaries (1965).

Fitted survival function to life-table data.

Imagery: 'power of man to avoid death'

Who coined the term 'force of mortality' ?

- Gompertz (1825) used the term 'intensity of mortality',
- Morabia (2005) $\Rightarrow$ Farr's "On prognosis" (1838).
- Farr probably got it from T R Edmonds
  - a political economist and actuary
  - neighbour who strongly influenced Farr's work
- Edmonds: 'force of mortality'
  - in *'italics and in quotes'* in first paragraph of his 1832 book
  - first person we know of to define as *force* of mortality

LIFE TABLES,

FOUNDED UPON

THE DISCOVERY

OF

A NUMERICAL LAW

REGULATING THE

EXISTENCE OF EVERY HUMAN BEING:

ILLUSTRATED BY

A NEW THEORY

OF THE

CAUSES PRODUCING HEALTH AND LONGEVITY.

By T. R. EDMONDS, B.A.
LATE OF TRINITY COLLEGE, CAMBRIDGE:
AUTHOR OF
"PRACTICAL MORAL AND POLITICAL ECONOMY."

LONDON:
PRINTED FOR
JAMES DUNCAN 37, PATERNOSTER ROW;
AND MAY BE HAD OF THE AUTHOR,
45, SOUTHAMPTON ROW, RUSSELL SQUARE.

M.DCCC.XXXII.

observed relation of Dying to Living, in given intervals of age. In constructing a Table of Mortality, the ordinary problem for solution is,—given, this relation for large intervals of age; required, to deduce and interpolate the relation of Dying to Living, corresponding to small intervals of age. In all Tables which have hitherto been published, this relation for annual intervals is continually varying. Now it is manifest, that the same principles which have led to the conclusion, that the variation is continued and *annual*, must lead to the conclusion, that the variation is monthly, and also to the conclusion, that the variation is diurnal, and even *momental*. It may be assumed, therefore, that all Tables of Mortality represent the relation of Dying to Living as changing continuously,—that this relation is never the same for any two successive instants of age. I have used the term "*force of mortality*," to denote this relation at any definite moment of age. It would evidently be improper to use this term to express the relation of Dying to Living in yearly intervals of age; for the force of mortality at the beginning, at the middle, and at the end of any year of age, are all different.

Edmonds' (and our) use of concept of a '*person-year*'

- "The force of mortality at any age is measured by the number of deaths in a given time, out of a given number constantly living. The given time has been here assumed to be one year, and the given number living to be one person; consequently, the force of mortality represents—the quantity of death in one year for a unit of life at the assumed age; or rather (since the force is changing continually) represents— the quantity of death on a unit of life which would occur by the action of this force continued uniform for the space of one year."
- Can use Poisson distr'n to derive $S(t) = \exp\{-\int_0^t h(u)du\}$
  - $\mu = \int_0^t h(u)$: *expected* number of events if always 1 (not necessarily *same*) individual at risk for the full $(0, t)$ interval.
  - $h(u)$ will vary over $(0, t)$, but can still view integral as the expected value of an (infinite) sum of Poisson r. v.'s.

Gompertz: words $\longrightarrow$ math'l form of $h(age)$?

"If the average exhaustions of a man's power to avoid death were such that at the end of equal infinitely small intervals of time, he lost equal portions of his remaining power to oppose destruction which lie had at the commencement of those intervals"

$\longrightarrow$ *"then at the age x his power to avoid death, or the intensity of his mortality might be denoted by* aq^x, *a and q being constant quantities"*

Modern-day notation: $\mu(x)$ *or* $h(age)$*:*

$$h(age) = h_0 \; e^{\beta \; age} \; ; \;\; \log[h(age)] = \beta_0 + \beta \; age$$

Gompertz translated, with help from Chiang

- Does Gompertz envision constant external force or threat?
- Is a man born with a certain power to avoid death, which is progressively exhausted with age?
- How can 'a power to resist death that decreases at a rate proportional to the power itself' lead to the stated form?

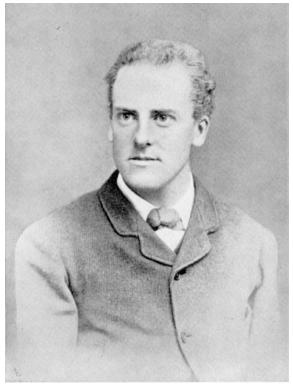
Since force of mortality $\mu(t)$ measures a person's susceptibility to death, Gompertz used the *reciprocal,* $1/\mu(t)$, as a measure of a person's *resistance* to (i.e., power to oppose) death".

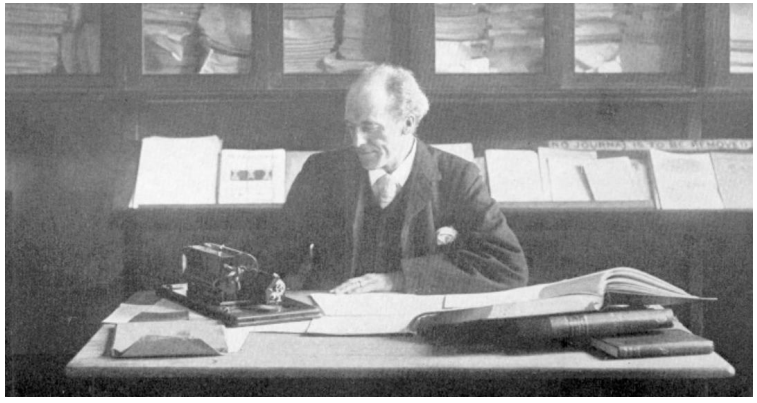
'constant loss of power to oppose'

$$\rightarrow \frac{d}{dt}\left\{\frac{1}{\mu(t)}\right\} = -k \times \left\{\frac{1}{\mu(t)}\right\}$$
$$\rightarrow \frac{d}{dt}\ \mu(t) = k \times \mu(t)$$

$\rightarrow$ $\mu(x)$ has form aq^x. $\rightarrow L_x = d \times (g)^{q^x}$.

Karl Pearson (1857-1936)





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...|.........|.........|.........|.........|.........|.........|.........|.........|......+
.1860......1870......1880......1890......1900......1910......1920......1930......1940
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Mediaeval imagery: the Dance of Death

Under roof of Spreuer Bridge in Lucerne, 67 paintings dating from 1626-1635 represent a "Dance of Death".

Death, represented as skeleton or as "Great Reaper" urges everybody to dance with him, i.e. to die. **Death makes no difference between old and young, churchmen and laymen**, ...

http://travelguide.all-about-switzerland.info/lucerne-spreuerbridge-dance-death.html

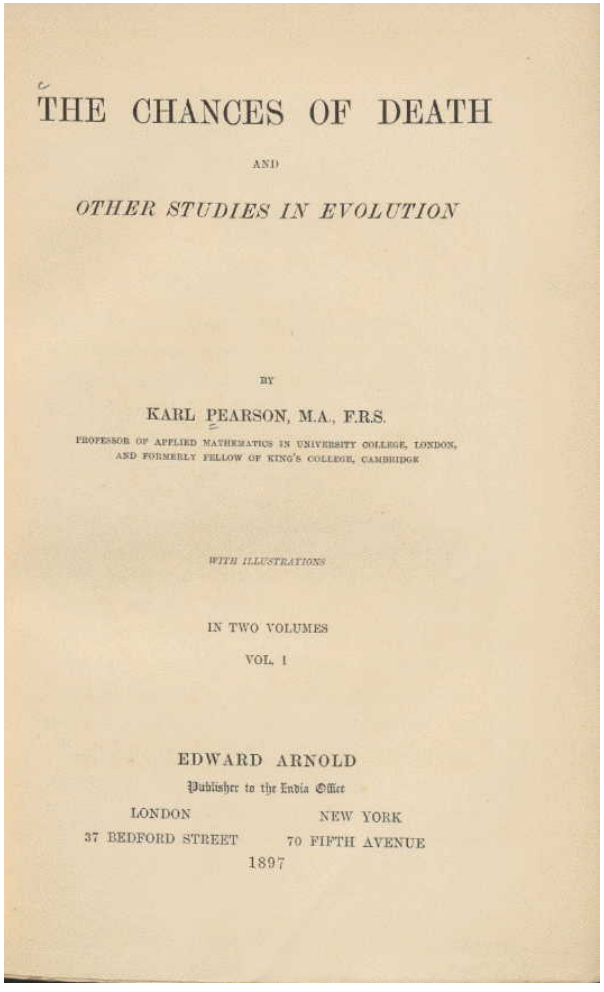
rich...


Wer auff sein Gut und Geld vertruwt
Und auff die Welt sein Wollust buwt
Ihr erins Werk sein Hoffnung richt
Der Tod mitt ihm ums Leben richt.

beautiful...

Seen by Pearson in 1875; motivation for his 1897 essay

Pearson's essay *The Chances of Death*, 1897



THE CHANCES OF DEATH

AND

OTHER STUDIES IN EVOLUTION

BY

KARL PEARSON, M.A., F.R.S.

PROFESSOR OF APPLIED MATHEMATICS IN UNIVERSITY COLLEGE, LONDON, AND FORMERLY FELLOW OF KING'S COLLEGE, CAMBRIDGE

WITH ILLUSTRATIONS

IN TWO VOLUMES

VOL. I

EDWARD ARNOLD

Publisher to the India Office

LONDON 37 BEDFORD STREET NEW YORK 70 FIFTH AVENUE

1897

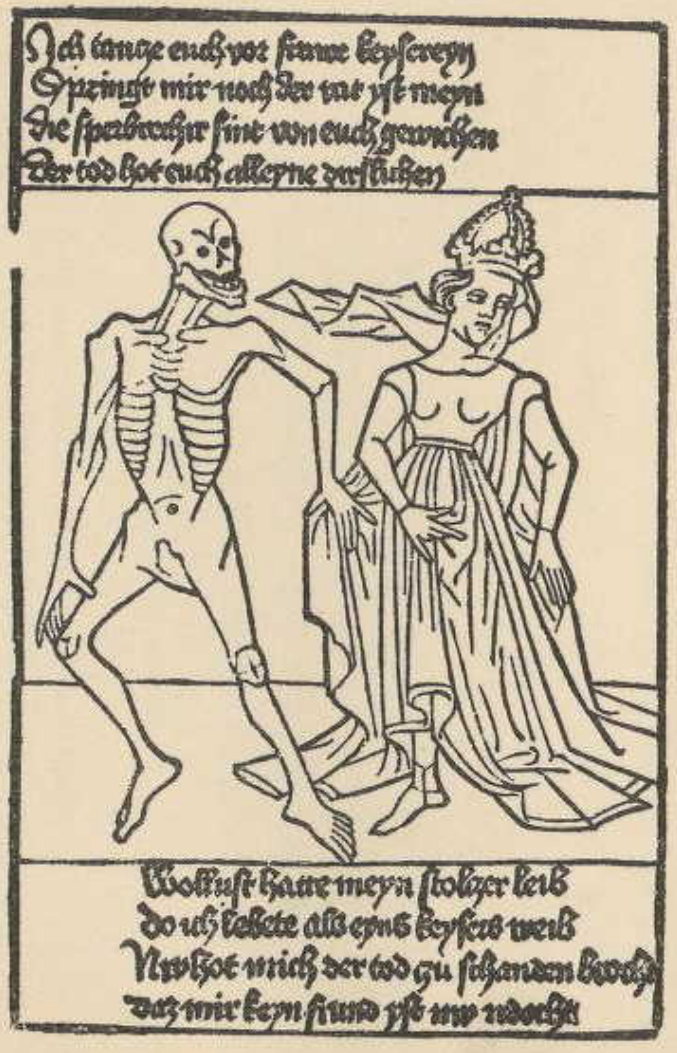
- Starts essay with mediaeval concept of death
  - Extract from Addison's *The Vision of Mirza*
  - 'Death has no calendar'
  - Death as one who 'obeys no rule of time,.. place, ..age,..sex,..household'
  - "Earliest Dance of Death of which fragments are still preserved to us, appears to be that on the wall of the cloister of a former nunnery at Klingenthal, near Klein-Basel; [...] language and costume of the designs suggest the first quarter of 14th century"

`www.dodedans.com/Ebasel-klingental.htm`

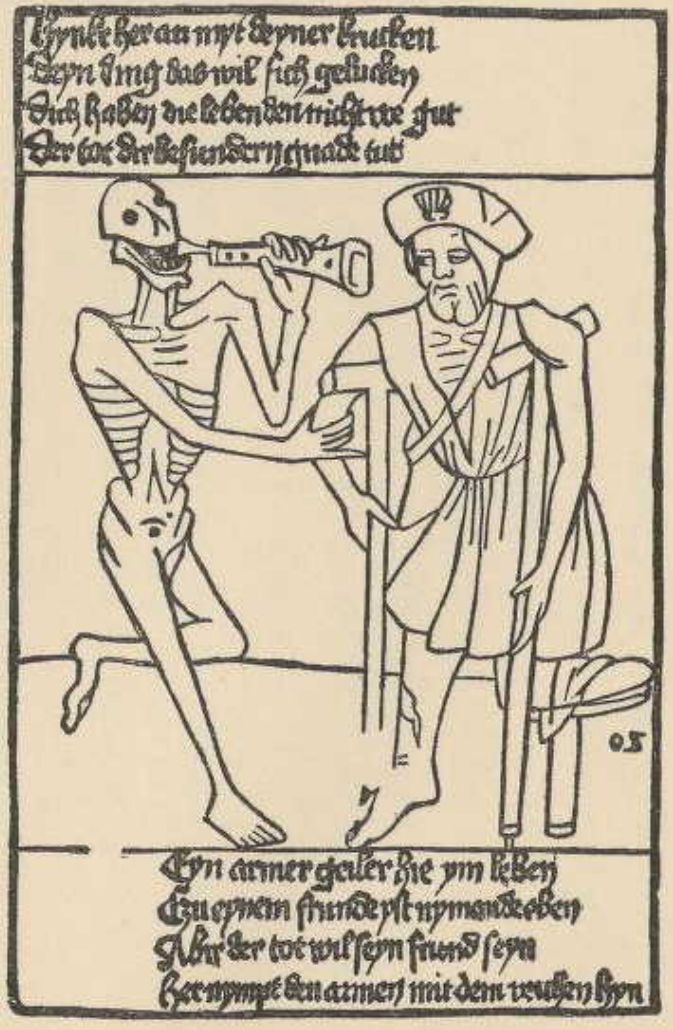
`www.dodedans.com/Eindex.htm`

“from Heidelberg Library, scarcely later than 1450...”

PLATE I.



Wollust hatte mein stolzer leib
do ich lebete als eyns keysers weib
Nu hot mich der tod zu schanden bracht
Das mir kein stund yst nw worden



Eyn armer geuler hie ym leben
Zu eynem stunde yst nymand eben
Aber der tot wil sein stund sein
Her nympt den armen mit dem reuben hin

HEIDELBERG BLOCK-BOOK.

Pearson's essay *The Chances of Death*, 1897

- 'Attempt to rise above the folk-conception of Chance as the chaotic, to the modern notion of Chance as obedience to law.'
- Evidence of 'law'/non-chaotic nature of Chance
  - Physical experiments, e.g. drawing counters from a bag
  - 'Frequencies more directly the product of Nature and less influenced by the hand of man' e.g. skull measurements
  - $\Rightarrow$ Death also governed by chance distributions, hence by 'obedience to law'
- Pearson a fan of frequency distributions
  - Data on distributions of deaths

From visual to statistical: Imagery of the marksman

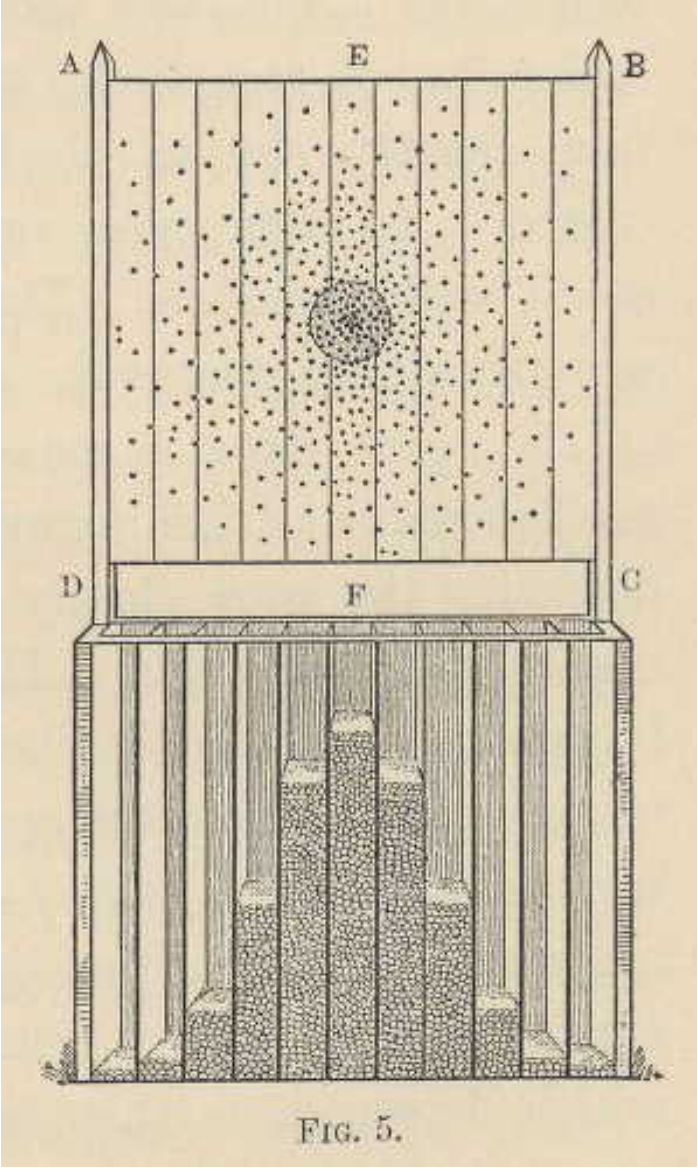


FIG. 5.

- Marksman fires bullets at target
- Standard deviation: ‘precision peculiar to a marksman or his weapon’
- Skew: peculiarity of marksman liable to miss more to left or to right of target

From visual to statistical: Imagery of the marksman

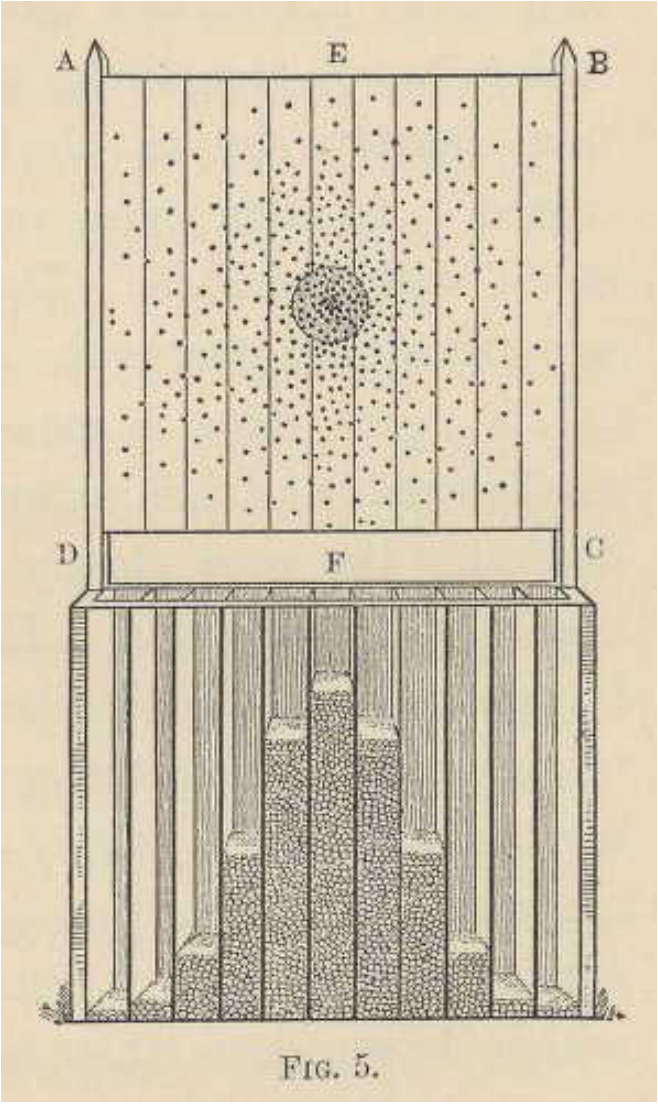
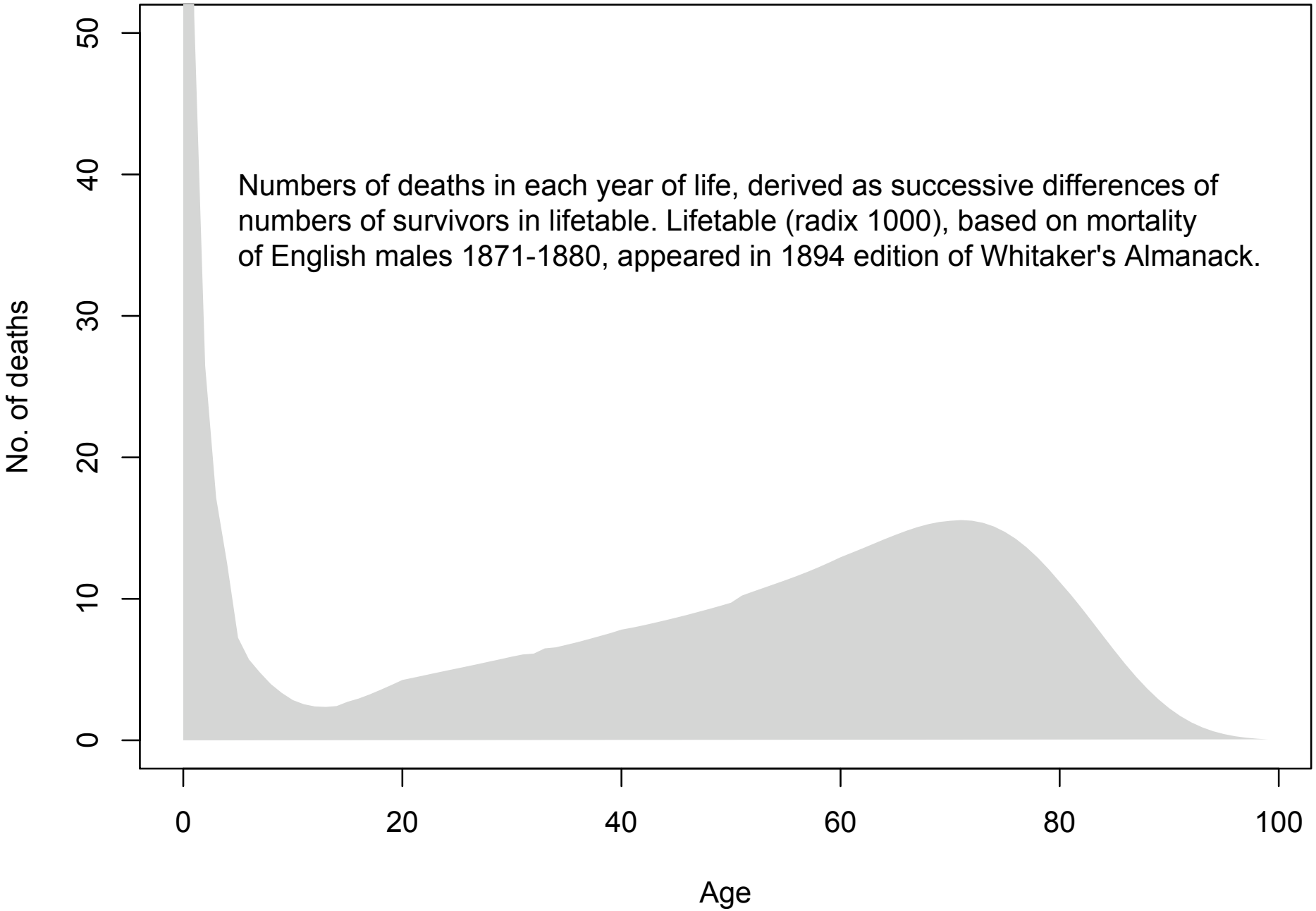


FIG. 5.

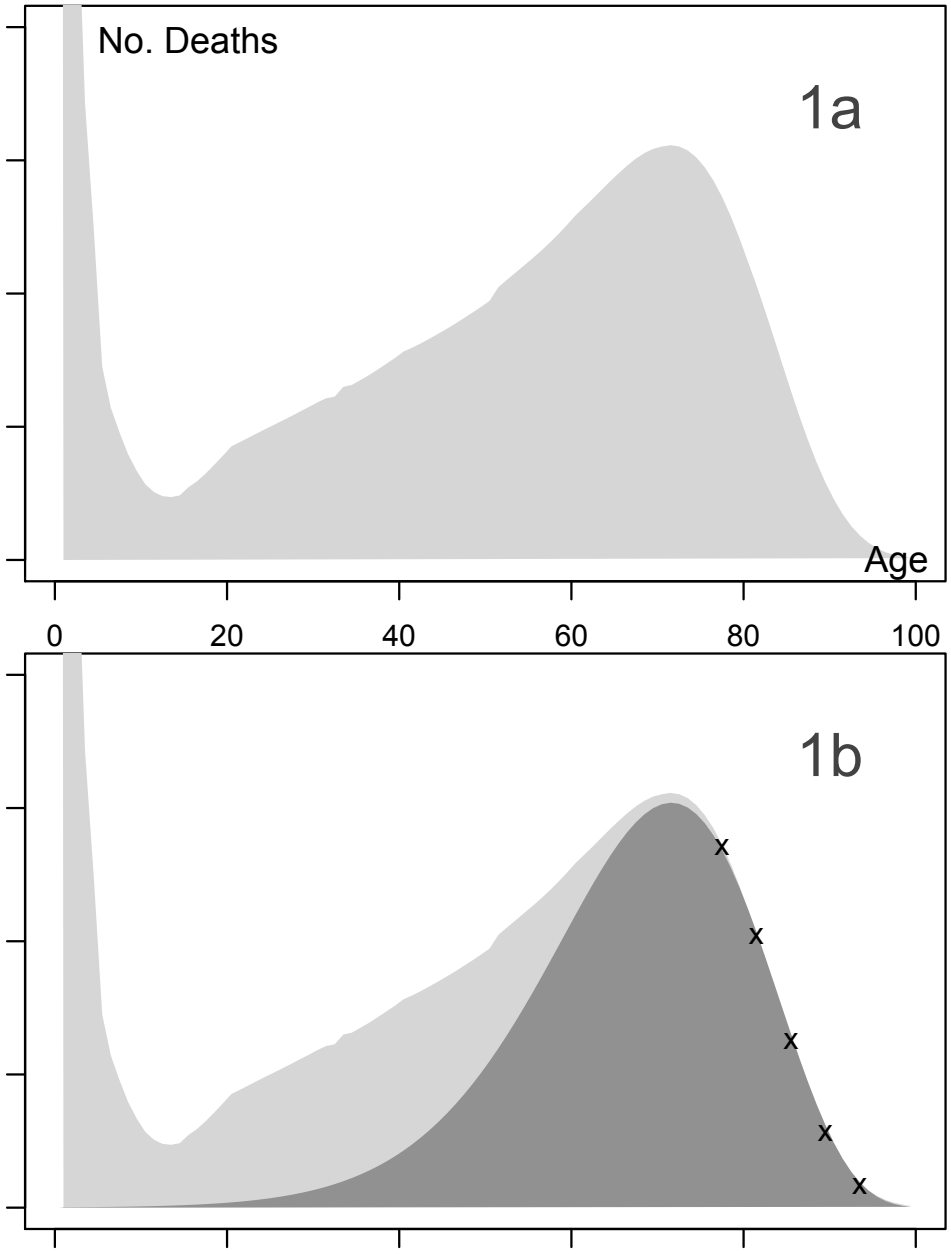
'Our ancestors were correct in supposing the frequency of death to be a chance distribution, but we now know that such a distribution follows regular laws, and this regularity we are able to picture to ourselves by thinking of Death as marksman with a certain skewness of aim and a certain precision of weapon'

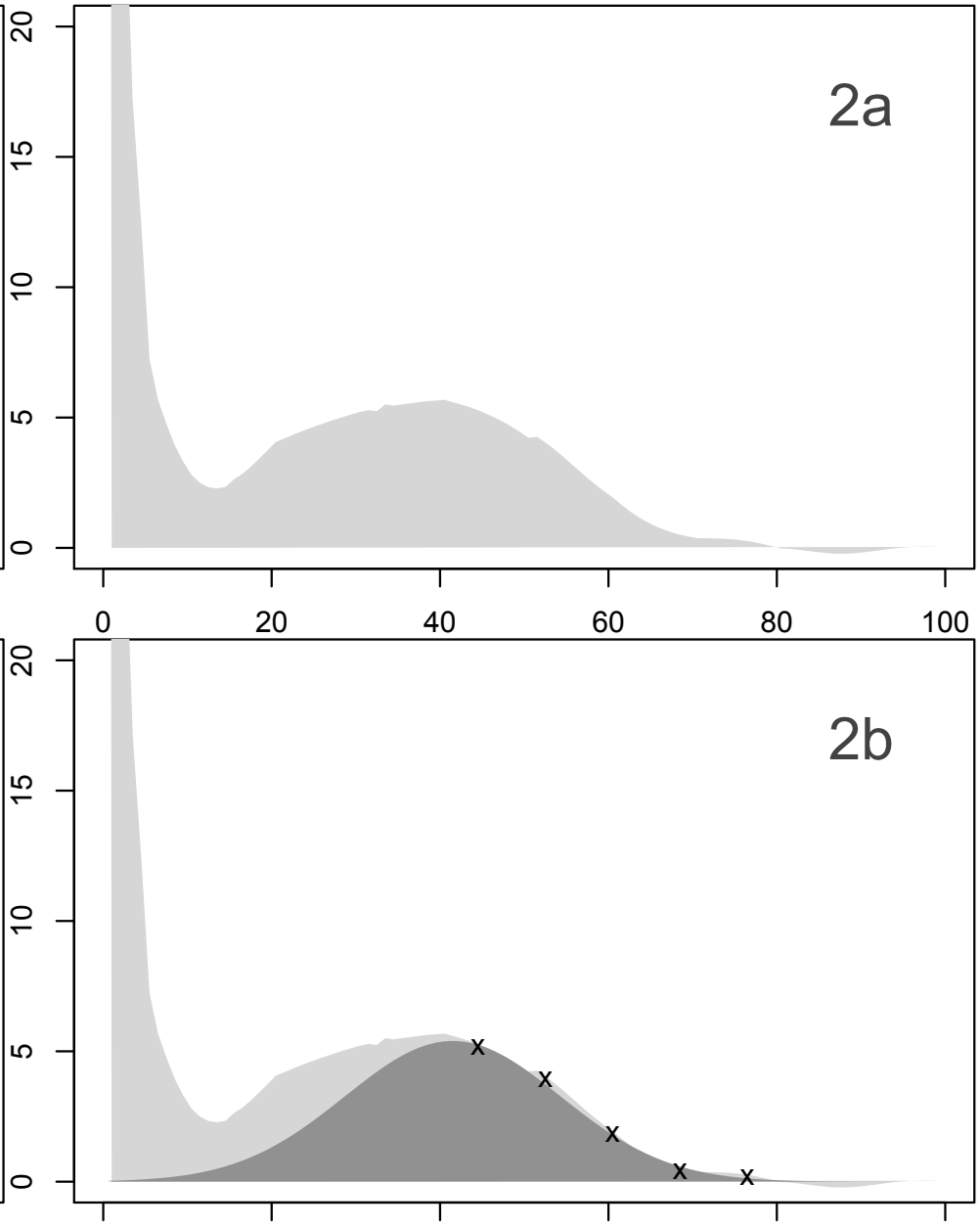
Idea: Force of mortality external to a person.

Pearson's data: frequency distribution of age at death

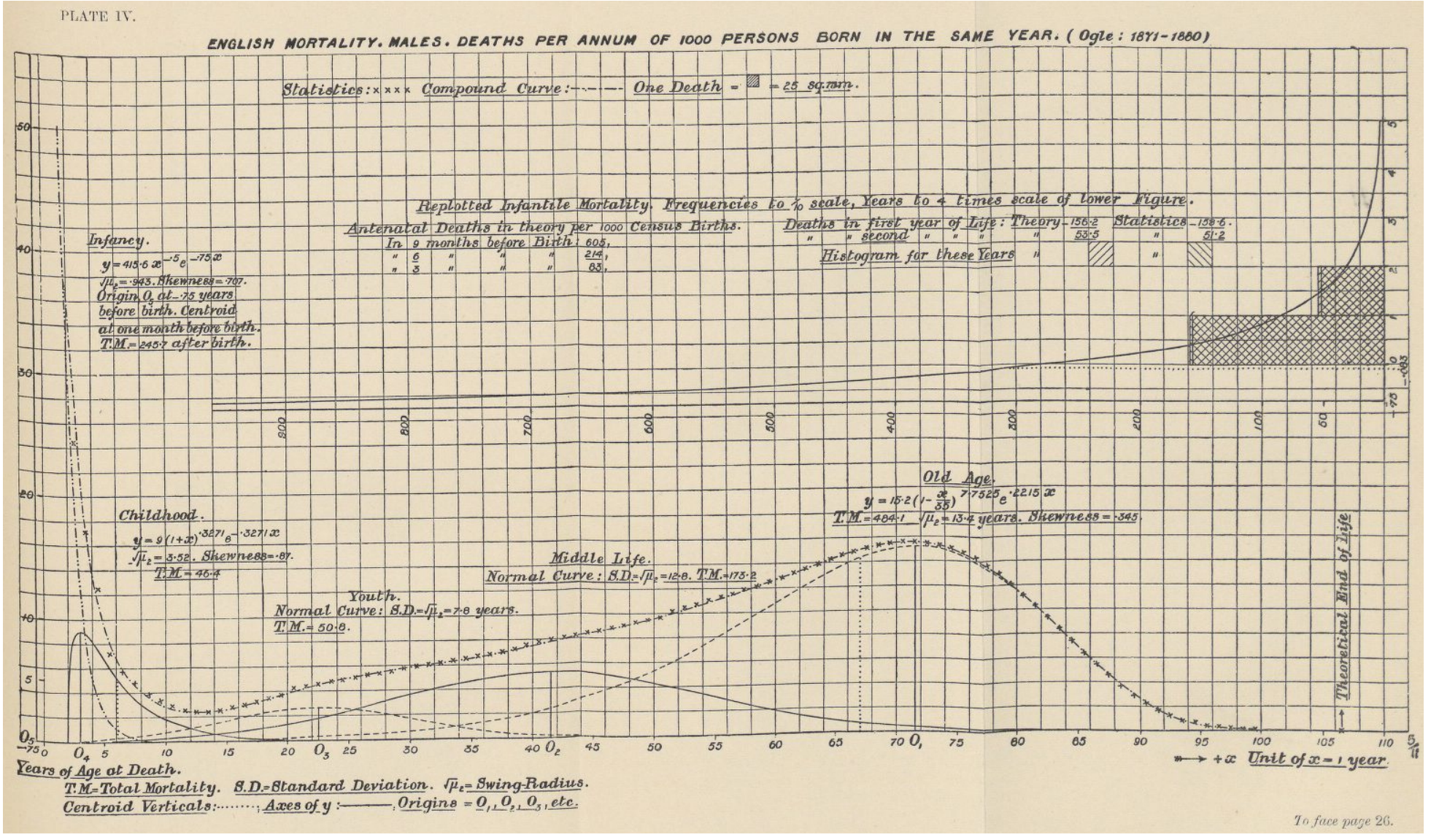


How Pearson fitted the 5-component mixture





The full 5-component mixture



Rendered by Pearson's wife, Maria Sharpe Pearson.

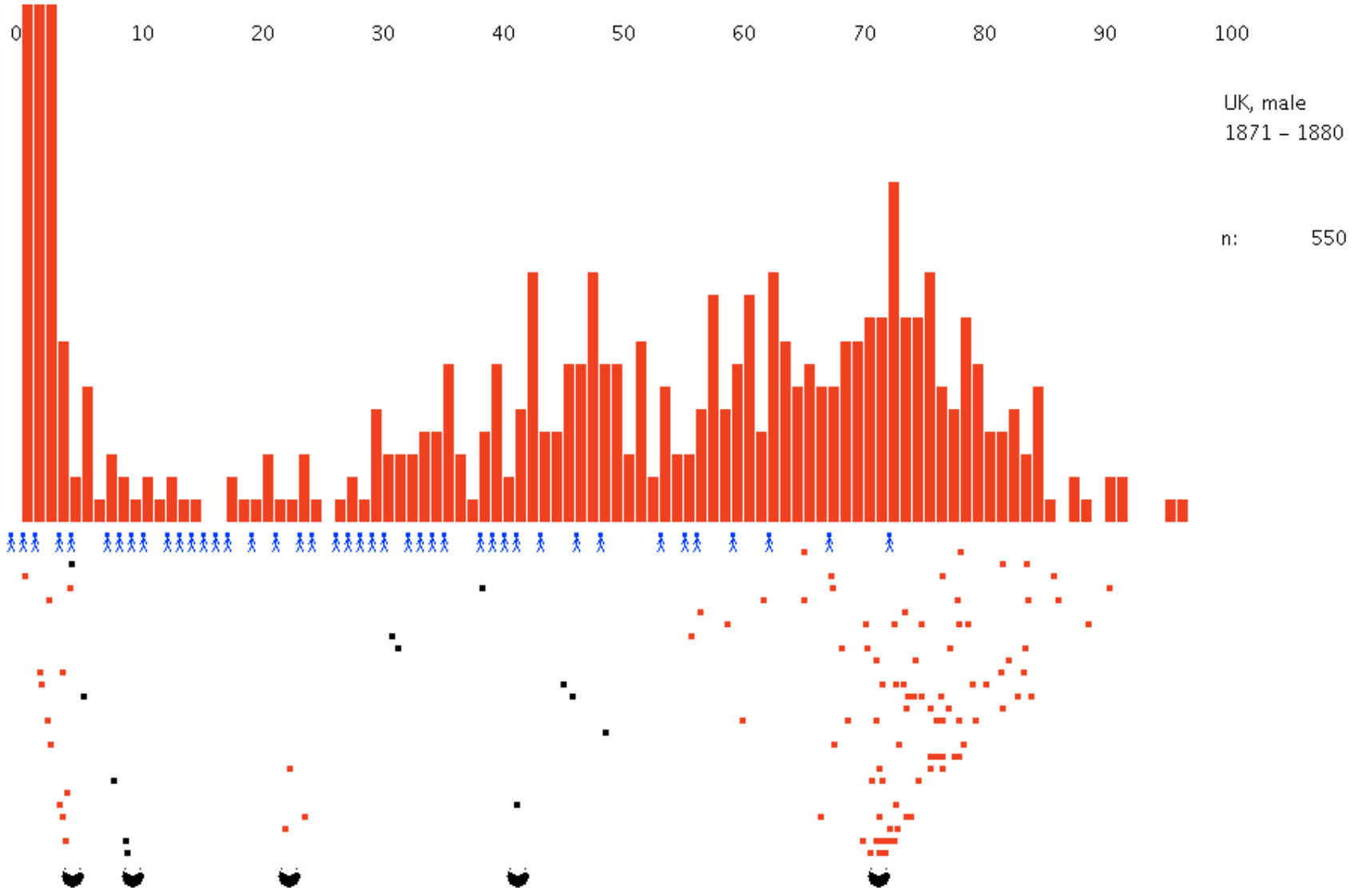
21st century animation of Pearson's Bridge of Life

Using hazard (intensity / force of mortality / incidence density) functions derived from Pearson's 5 curves

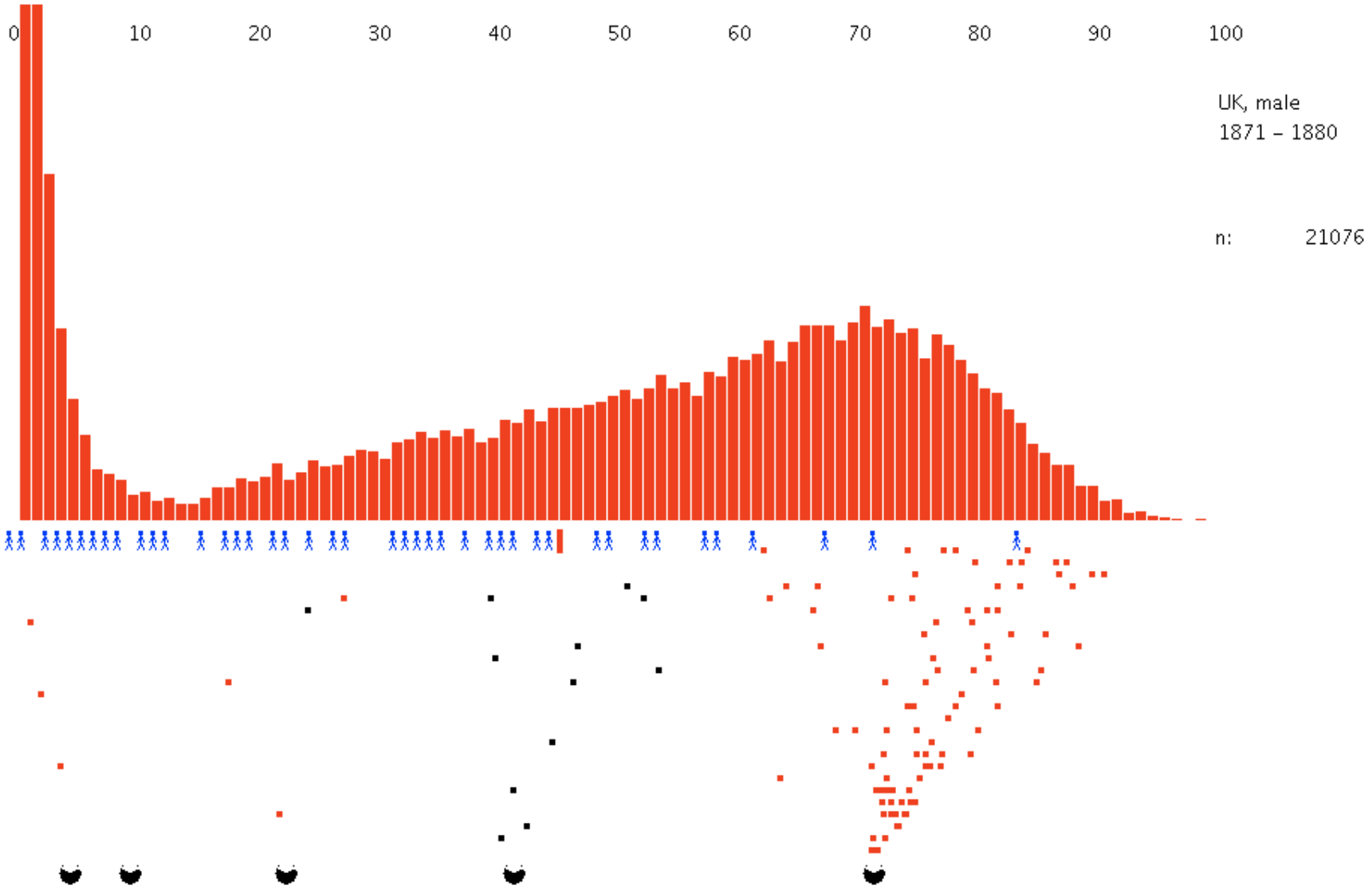
http://www.biostat.mcgill.ca/hanley/BridgeOfLife/

WARNING: Moving images shown

21st century animation of Pearson's Bridge of Life



21st century animation of Pearson's Bridge of Life



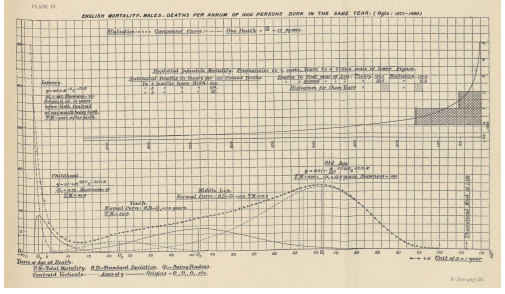
Cultural imagery and statistical models of the force of mortality: Discussion

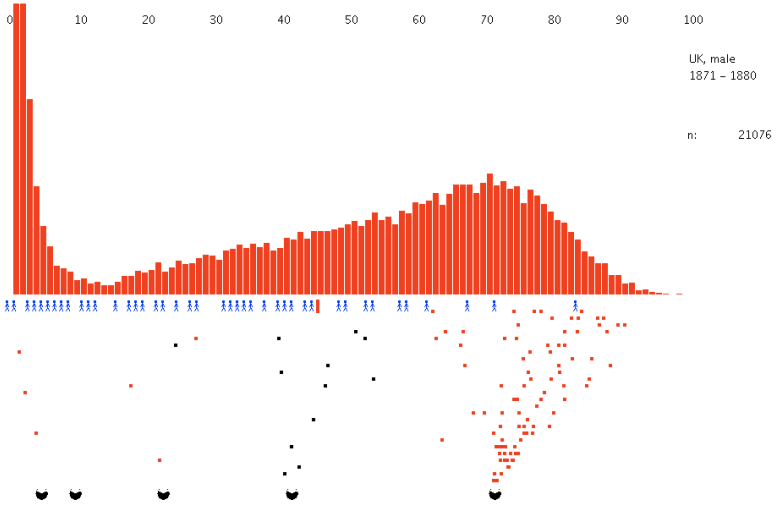
- Force of mortality and its components/derived quantities can be difficult to explain
- Three teachers give human form to components of force of mortality, $h(x) = \frac{f(x)}{1-F(x)}$

Teacher	Allegory	Conceptual model	Statistical model
Addison, 1711	Yes	External/Trap-doors	-
Gompertz, 1825	-	Internal/Body wears out	Smooth hazard/survival function
Pearson, 1897	-	External /Marksman	Mixture of frequency functions

- Three teachers $\longrightarrow$ 21st century computer animation

Force of Mortality and Bridge of Life





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