NOTES ON THE THEORY OF PROBABILITIES.

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15. The problem which we are about to consider is one of the most fruitful in the theory of probabilities, as out of it grow the theory of errors, the theory of chance distribution, the law of averages, and the estimation of the probability that an observed concurrence of events is the result of a law of nature.

To find the probability that an event of which the probability on a single trial is p *will happen* s *times on* n *trials.*

The probability that it will fail on every trial is $(1-p)^n$, $1-p$ being the probability that it will fail on any single trial.

The probability that it will happen on the first trial and fail on the $n-1$ following ones is $p(1-p)^{n-1}$. But as the single event is as likely to occur on the 2d, 3d, ... nth trial as on the first, the probability that it will occur just once is $np(1-p)^{n-1}$.

The probability that the event will occur on the first two trials and fail on the $n-2$ subsequent ones is $p^2(1-p)^{n-2}$. But the two events can equally occur on the $(1, 3)$, $(1, 4)$ . . . . . $(1, n)$, or the $(2, 3)$, $(2, 4)$, &c. trials; in fact there will be $\overset{2}{\underset{n}{C}}$ pairs of trials on which the two events can occur, so that $\overset{2}{\underset{n}{C}} p^2(1-p)^{n-2}$ is the probability that it will occur twice.

By a process of reasoning exactly like the last, we find the probability that it will occur s times to be

$$(1) \qquad P_s = \overset{s}{\underset{n}{C}} p^s (1-p)^{n-s},$$

which is the $(s+1)$st term in the development of the binomial $[(1-p)+p]^n$. The sum of the probabilities of all the possible results of the n trials is therefore 1, as it ought to be.

As an example to elucidate the above, suppose that a cent is so formed that a head is twice as likely to be thrown as a tail, so that the probability of the former on each throw is $\frac{2}{3}$. If the coin is thrown four times, the results of the four throws may be as follows. After each separate result is written the fraction expressing the probability of that result.

No heads, $tttt\ \frac{1}{3^4} = \frac{1}{81}$,
1 head, $httt$, $thtt$, $ttht$, $ttth$, each $\frac{2}{81}$; $\times\ 4 = \frac{8}{81}$,
2 heads, $hhtt$, $htht$, $htth$, $thht$, $thth$, $tthh$, $6 \times \frac{4}{81} = \frac{24}{81}$,
3 heads, $hhht$, $hhth$, $hthh$, $thhh$, $= \frac{32}{81}$,
4 heads, $hhhh = \frac{16}{81}$.

If we supposed heads as likely to be thrown as tails, we should find these probabilities to be $\frac{1}{16}$, $\frac{4}{16}$, $\frac{6}{16}$, $\frac{4}{16}$, $\frac{1}{16}$, respectively. The result would evidently be the same if we supposed that four coins were thrown from a box together.

16. To resume the general discussion, let us see what value of s is the most probable. This value we will determine by the condition that its probability must be greater than that of the next smaller number, and also greater than that of the next greater number, or

$$\overset{s}{\underset{n}{C}} p^s (1-p)^{n-s} > \overset{s-1}{\underset{n}{C}} p^{s-1} (1-p)^{n-s+1};$$

$$\overset{s}{\underset{n}{C}} p^s (1-p)^{n-s} > \overset{s+1}{\underset{n}{C}} p^{s+1} (1-p)^{n-s-1}.$$

Since $$\overset{s}{\underset{n}{C}} = \frac{n-s+1}{s} \overset{s-1}{\underset{n}{C}}; \qquad \overset{s+1}{\underset{n}{C}} = \frac{n-s}{s+1} \overset{s}{\underset{n}{C}};$$

we have from the division of the first inequality by $\overset{s-1}{\underset{n}{C}} p^{s-1} (1-p)^{n-s}$,

$$1 - p < \frac{n-s+1}{s} p, \qquad \text{which gives } s < p(n+1);$$

and from the division of the second inequality by $\overset{s}{\underset{n}{C}} p^s (1-p)^{n-s-1}$ we have

$$1 - p > \frac{n-s}{s+1} p, \qquad \text{which gives } s > p(n+1) - 1;$$

s is therefore the greatest whole number in $p(n+1)$. If s and n are very large numbers, we have very nearly

$$(2) \qquad \frac{s}{n} = p.$$

It follows, therefore, that in a great number of trials events are more likely to occur a number of times proportional to their respective probabilities than any other number. Thus if a cent is thrown one hundred times, heads are more likely to be thrown fifty than any other single number of times. But it must not be supposed that they are therefore more likely to be thrown fifty times than not, for it is *almost* as likely to be thrown 49 times or 51 times as fifty times. The chances that it would be thrown *exactly* fifty times would be quite small, because there are so many other numbers that might be thrown.

17. Another deduction from the expression (1) is the following: however small the probability of an event on a single trial, by increasing the number of trials we can render the probability that the event will occur at least once as great as we please. For the probability that it will fail on every one of the n trials being $(1-p)^n$; however small p may be, we may make n so great that $(1-p)^n$ shall be as small as we please.

18. Suppose now that n is infinitely great, and p infinitely small, and that $np = a$, a being a finite quantity. We may then put $n = n - 1 = n - 2$, &c. We shall then have, while s is finite,

$$\overset{s}{\underset{n}{C}} p^s = \frac{n^s p^s}{s!} = \frac{a^s}{s!},$$

$$(1-p)^{n-s} = (1-p)^n = (1-p)^{\frac{a}{p}} = e^{-a},$$

e being the Napierian base. Substituting these values in (1) we obtain for the probability that the event will occur s times

(3) $$P_s = \frac{\alpha^s e^{-\alpha}}{s!}.$$

The probability that the event will fail, is therefore $e^{-\alpha}$; that it will occur once only, $\alpha e^{-\alpha}$; twice only, $\frac{\alpha^2}{2} e^{-\alpha}$, &c.

The sum of this series of probabilities continued to infinity is

$$e^{-\alpha}\left(1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \&c.,\right) = e^{-\alpha} \,.\, e^{\alpha} = 1, \text{ as it ought to be.}$$

19. We may apply this equation to the determination of the probability that, if the stars were scattered at random over the heavens, any small space selected at random would contain s stars. Let N be the whole number of stars, h the number of units of space in the heavens, then $\frac{N}{h}\,d\,h$ may be taken to represent the infinitely small probability that the infinitely small space $d\,h$ contains a star. Moreover, if l represents the extent of space selected at random which we consider, we may consider the examination of each $d\,h$ as a trial, and the number n of trials will then be $\frac{l}{d\,h}$. The value of α will then become $\frac{N}{h}\,l$, and by substitution in (3) we have

(4) $$P = \frac{N^s\, l^s\, e^{-\frac{N l}{h}}}{h^s\, s!}$$

for the probability that the space l contains s stars. Suppose, as a numerical example, that l is a square degree, and $N = 1500$, which is about the number of stars of the fifth and higher magnitudes; $s = 6$. We then have $\frac{l}{h} = \frac{1}{41253}$; by the substitution of these values in (4) we shall have the probability that any square degree selected at random in the heavens contains six stars. Multiplying this probability by 41253, the number of square degrees in the whole heavens, we obtain the probability that, *if the heavens were divided at random into square degrees, some one of those square degrees would contain six stars.* This probability we find to be

$$\frac{1500^6}{41253^5 . 6!} \cdot e^{-.08686} = .000000128.$$

This, however, is evidently rather smaller than the probability that six stars should be found so near together that a square degree could be fitted on so as to include them.

Mr. MITCHELL committed an error in his solution, the effect of which, if I mistake not, is to make his probability too great. His general method is, however, better applicable to this particular problem than that given above, but as there is a margin of vagueness and uncertainty about the problem in question, so that the answer does not admit of being expressed in exact numbers without an excessively complicated process of reasoning, I have preferred to deduce an approximate solution from the general formulæ, to be used in so many more problems.

20. Let us now consider Prof. FORBES's objections to the above results of the calculus of probabilities.

He scattered paint from a brush upon a wall, and found double and triple spots and groups innumerable. This is about as decisive as an attempt to disprove the Pythagorean proposition by measuring the squares described on a triangle without knowing whether it had or had not a right angle, and finding that one square was not equal to the sum of the others. As a mathematician would answer this objection by saying that his result only proved that he had either made a mistake in his measurements, or had not measured a right triangle; so Prof. FORBES's result only proves that either he was mistaken as to the marked character of the groupings, or that the proximity of the components of each group was the effect of their positions being determined *by the action of the same cause*, which is all that the theory of probabilities claims for the Pleiades. The latter supposition is by no means improbable, because a group of spots might be formed by the breaking up of a drop after it had left the brush.

21. Prof. Forbes remarked that an exactly uniform distribution of stars would not be expected as the result of a random distribution. In this he is correct; and as it is an interesting problem, we shall here determine what law a random distribution may be expected to follow. It may appear paradoxical to assert that the results of chance can be expected to follow any law; but such is really the case, and the formula (3) determines the law. As an example, suppose that the heavens are divided into 1500 equal portions, and that 1500 stars are distributed at random, or, to speak with more philosophical accuracy, that the causes which determine the position of each separate star are entirely independent of those which determine the position of any other. Then by reasoning as in § 19 we find $\alpha = 1$; and by formula (3) the probability that a unit of space selected at will contains no star, will be $\frac{1}{e} = \frac{1}{2.718\ldots}$*; one star, $\frac{1}{e}$; two stars, $\frac{1}{2e}$; three stars, $\frac{1}{2.3e}$; &c. If we then select the whole 1500 units we ought to expect the number which would be found to contain the several numbers of stars to be somewhere near 1500 multiplied by the respective probabilities, or

about	$\frac{1500}{e} = 552$	portions	containing	no star,
"	$\frac{1500}{e} = 552$	"	"	1 star,
"	$\frac{1500}{2e} = 276$	"	"	2 stars,
"	$\frac{1500}{2.3e} = 92$	"	"	3 stars,
"	$\frac{1500}{4!e} = 23$	"	"	4 stars,
"	$\frac{1500}{5!e} = 4(+)$	"	"	5 stars,
"	$\frac{1500}{6!e} = 1$	"	"	6 stars,

* The acute reader will perceive that the solutions in §§ 19 and 21 are those of a problem slightly (though not materially) different from that actually propounded.

and it would be quite improbable (about 1 chance to 8) that any space would be found to contain more than six stars.

If any one wishes an experimental illustration of the principle let him take a pint of rice, color a hundred grains of it black, mix the black grains thoroughly with the remainder, and stir the mixture till he finds six or eight of the black grains to form a group by themselves. Before this result arrives he will in all probability be willing to admit that should he ever see such a group in such a mixture, he would not believe that it was formed by indiscriminate mixing.

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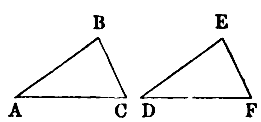
CHAPTER VI.

THE PYTHAGOREAN PROPOSITION.

91. *Theorem.* If a triangle has one side and the adjacent angles equal respectively to a side and the adjacent angles in another triangle, the two triangles are equal. — *Proof.* Let us suppose that, in the triangles A B C and D E F, we have the side A B equal to the side D E, the angle at A equal to the angle D, and that at B equal to that at E. Let us imagine the triangle D E F to be laid upon A B C in such a manner as to place E upon B, and D upon A, which can be done because A B is equal to D E. Now, as the angle A is equal to D, the line D F will run in the same direction as A C, and, as it starts from the same point, will coincide with it. Also, since the angle B is equal to E, the line E F will coincide with B C. Whence, by article 90, the triangles are equal.



92. *Theorem.* The opposite sides of a parallelogram are equal. — *Proof.* Article 90 gives us the only test of geometrical equality. So that in order to prove this theorem we must show that in a parallelogram like A B C D, A B may be made to coincide with D C, and B C with A D. And this would evidently be done if we could show that the triangle A B C is equal to A D C. But in these triangles the line A C is the same, and by article 87 the adjacent angles A C B and C A B are equal to the adjacent angles C A D and A C D; whence, by article 91, the two triangles are equal, and A D is equal to B C, and A B equal to D C.

Fig. B